Exam Statistical Methods in Physics
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Before you start, read the following:

- Write your name and student number on top of each page of your exam;
- Illegible writing will be graded as incorrect;
- *Good luck!*

Problem 1 (15 points)

Read the following statements carefully, and indicate if they are true or false:

(a) The marginal distributions of a multi-dimensional PDF are normalized.

(b) The PDF of x and y is given by $f(x, y) \propto \sin(xy)$ with $0 \leq x, y \leq \pi$, so x and y are independent.

(c) The covariance of x and y, $cov(x, y) = 0$, so x and y are uncorrelated.

(d) In hypothesis testing, the main motivation in choosing the critical region is to make it as large as possible.

(e) A consistent estimator can be biased.

(f) The likelihood of a parameter t, $\mathcal{L}(\vec{x}|t)$, for a set of measurements $\vec{x}$ defines the PDF for t.

(g) Bayes' theorem relates the conditional probabilities $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$.

(h) For x : Normal$(x; \mu, \sigma^2)$, the normalized variable $t = \frac{\overline{x}_n - \mu}{s/\sqrt{n}}$ with $\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2$ has a Student-t distribution with $N-1$ degrees of freedom.

(i) By carefully designing a hypothesis test, loss α and contamination β can always be made arbitrarily small.

(j) If l_i are standard Normal distributed, then $\sum_{i=1}^{s} l_i^2$ has a χ^2 distribution with s degrees of freedom.

(k) $N\, I_1(\theta)\, V_{\hat{\theta}} \geq 1$.

(l) If P and Q have a multi-variate Normal distribution with correlation ρ and variances σ_P and σ_Q, you can always apply a coordinate transformation and define U and V such that they are indepdendent.

(m) When testing how well your prediction fits the data, you find a χ^2 of 18 for 40 degrees of freedom. This is expected about half of the time.

(n) After a succesfully minimizing χ^2, the average of the residuals $\langle (x - f)/\sigma \rangle = 0$.

(o) The sample mean is always a sufficient statistic of the mean of the underlying distribution.

(p) Doubling the amount of information requires quadrupling the number of measurements.

Problem 2 (15 Points)

A cucumber growers asks for your help to optimize his profit. Consumer cucumbers have to be straight, as defined by the parameter c. The grower has measured c for 100 of his cucumbers:

0.74	1.43	1.82	2.23	2.77	3.23	3.56	4.18	4.65	5.42
0.88	1.50	1.84	2.45	2.86	3.23	3.59	4.20	4.75	5.65
0.91	1.53	1.90	2.48	2.89	3.33	3.63	4.22	4.81	5.66
0.92	1.57	1.90	2.49	2.94	3.40	3.74	4.27	4.82	6.32
1.08	1.58	1.97	2.52	2.99	3.41	3.78	4.34	4.92	6.41
1.24	1.65	2.05	2.59	2.99	3.42	3.85	4.40	4.93	7.20
1.25	1.65	2.08	2.59	3.03	3.46	3.88	4.44	4.95	7.28
1.30	1.68	2.12	2.61	3.13	3.53	3.95	4.51	4.95	7.54
1.34	1.78	2.15	2.70	3.19	3.53	4.06	4.57	5.07	12.18
1.37	1.79	2.18	2.73	3.21	3.54	4.13	4.64	5.20	19.31

(a) Make an equal-width histogram for the grower's data. Motivate your choice of the range and the number of bins.

(b) You suspect that c is described by the Rayleigh distribution:

$$f(c) = \frac{c}{\sigma^2} e^{-\frac{1}{2}\left(\frac{c}{\sigma}\right)^2}$$

What is the distribution mean $E[c]$? Calculate the sample mean $\overline{c}$ using your histogram. What is your estimate for $\hat{\sigma}$? *Hint:* $\int_0^\infty x^2 e^{-x^2} dx = \sqrt{\pi}/4$.

(c) What is the probability $\mathbb{P}(a \leq c \leq b)$ to find c between a and b for a given σ? Recall that $2x\, dx = d(x^2)$.

(d) Should you discard the two measurements at 12.18 and 19.31? Motivate why (not)?

(e) Perform a χ^2-test to check whether your assumption that the data is described by the Rayleigh distribution is true. Ignore bins with zero counts. Discuss your result.

Problem 3 (15 Points)

When you posses a car with an average CO_2 emission η below 110 mg/km you do not have to pay road tax in the Netherlands. A certain type of car, the Belchfire Runabout, is tested by the government agency RDW to see if it meets this criterium. It is known that the measured emission is described by a Normal distribution $N(\mu, \sigma^2)$ with μ and σ both unknown.

(a) As usual, the RDW measures the emission for five Runabouts, which yield the following (independent) results (all in mg/km):

$$\eta_i = \{110.48, 106.02, 106.42, 108.62, 106.27\}.$$

Calculate the sample mean $\overline{\eta}$ and sample variance s_η. To be tax-exempt, the *lower* 95% confidence interval for the true emission μ is required to be entirely below 110 mg/km. Does this car type meet the tax-exempt criterium?

(b) The Belchfire company makes adjustments to the design of the Runabout in an attempt to reduce the emission from the (assumed) current emission of 108 mg/km to 105 mg/km (or less). The emission of their redesigned car is measured n times. It is known that the standard deviation of a single measurement with their set up is 2 mg/km. Assume that the measurements are independent. Give the distributions of the mean of the measured emission for the old and new engine (with n as a parameter).

(c) Design a test to see whether the new design meets Belchfire's goal with 99% confidence. How large should n be to have a type-II error (contamination) of less than 1%?

(d) Should the Belchfire company require a higher level of confidence then the RDW? Would it be wise to test more cars than the RDW does? Why (not)?

(e) In an attempt to be as sure as possible, Belchfire has made 100 measurements of the emission of the Runabouts. The engineers are convinced that the emission is 105 mg/km with 99% confidence. The car is tested by the RDW and *not* approved for tax exemption. Is this possible? Explain and give an estimate for the probability that this occurs. You may assume that the RDW hasn't changed its testing procedure. Should you suspect there is something wrong with the testing equipment at the RDW?

Problem 4 (15 Points)

The main cable for a cable car consist of 16 thin cables wound together. The thin cables have a failure rate of $\lambda = 4 \times 10^{-5}$ per meter length.

(a) Give the distribution for the number of failures for a cable of length L. What is the probability that this cable breaks?

(b) The main cable is supported by a tower every 300 m. What is the distribution of the number of broken thin cables in a 300 m long stretch of the main cable?

(c) The total length of the cable is 3 km. Safety standards require that that the main cable has no more than 10 cable breaks, and under no condition two or more breaks in a single section. What is the probability that the main cable fails to meet this criterium?

Problem 5 (15 Points)

In the 18^{th} century Geroges-Louis Leclerc, Compte de Buffon (1707-1788) found an amusing way to approximate the number π using probability theory and statistics. Buffon had the following idea: take a needle and a large sheet of paper, and draw horizontal lines that are a needle-length apart. Throw the needle a number of times (say n times) on the sheet, and count how often it hit one of the horizontal lines. Say this number is s_n, then s_n is the realization of a $Bin(n,p)$ distributed random variable. Here p is the probability that the needs hits one of the horizontal lines.

(a) Assume that the length of the needle is 1 and that the lines are parallel to the x-axis. Argue that Z, the distance from the center of the needle (X, Y) to the first line with smaller y is given by a uniform distribution with $U(0, 1)$. Also argue that ϕ, the ange between the needle and the positive x-axis has a $U(0, \pi)$ distribution and that Z and ϕ are independent.

(b) Show that the needle hits a horizontal line when

$$Z \leq \frac{1}{2}\sin\phi \quad \text{or} \quad 1 - Z \leq \frac{1}{2}\sin\phi$$

(c) Show that the probability that the needle will hit one of the horizontal lines is equal to $2/\pi$.

(d) Show that $T = \frac{2n}{s_n}$ is the maximum likelihood estimator for π.

Table 1: Integral of the Standard Normal distribution: $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 dx}$.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.50	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.60	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.70	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2: Quantiles of the Chi-squared distribution: $\int_0^x \chi^2(NDF)dx = \alpha$

NDF; α	0.005	0.010	0.025	0.050	0.900	0.950	0.975	0.990	0.995
1	0.000	0.000	0.001	0.004	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	118.498	124.342	129.561	135.807	140.169

Table 3: Quantiles of Student's t-distribution: $\int_{-\infty}^{x} Stud(NDF)dx = \alpha$

NDF; α	0.550	0.600	0.680	0.750	0.900	0.950	0.975	0.990	0.995
1	0.158	0.325	0.635	1.000	3.078	6.314	12.706	31.821	63.657
2	0.142	0.289	0.546	0.816	1.886	2.920	4.303	6.965	9.925
3	0.137	0.277	0.518	0.765	1.638	2.353	3.182	4.541	5.841
4	0.134	0.271	0.505	0.741	1.533	2.132	2.776	3.747	4.604
5	0.132	0.267	0.497	0.727	1.476	2.015	2.571	3.365	4.032
6	0.131	0.265	0.492	0.718	1.440	1.943	2.447	3.143	3.707
7	0.130	0.263	0.489	0.711	1.415	1.895	2.365	2.998	3.499
8	0.130	0.262	0.486	0.706	1.397	1.860	2.306	2.896	3.355
9	0.129	0.261	0.484	0.703	1.383	1.833	2.262	2.821	3.250
10	0.129	0.260	0.482	0.700	1.372	1.812	2.228	2.764	3.169
11	0.129	0.260	0.481	0.697	1.363	1.796	2.201	2.718	3.106
12	0.128	0.259	0.480	0.695	1.356	1.782	2.179	2.681	3.055
13	0.128	0.259	0.479	0.694	1.350	1.771	2.160	2.650	3.012
14	0.128	0.258	0.478	0.692	1.345	1.761	2.145	2.624	2.977
15	0.128	0.258	0.477	0.691	1.341	1.753	2.131	2.602	2.947
16	0.128	0.258	0.477	0.690	1.337	1.746	2.120	2.583	2.921
17	0.128	0.257	0.476	0.689	1.333	1.740	2.110	2.567	2.898
18	0.127	0.257	0.476	0.688	1.330	1.734	2.101	2.552	2.878
19	0.127	0.257	0.475	0.688	1.328	1.729	2.093	2.539	2.861
20	0.127	0.257	0.475	0.687	1.325	1.725	2.086	2.528	2.845
21	0.127	0.257	0.475	0.686	1.323	1.721	2.080	2.518	2.831
22	0.127	0.256	0.474	0.686	1.321	1.717	2.074	2.508	2.819
23	0.127	0.256	0.474	0.685	1.319	1.714	2.069	2.500	2.807
24	0.127	0.256	0.474	0.685	1.318	1.711	2.064	2.492	2.797
25	0.127	0.256	0.473	0.684	1.316	1.708	2.060	2.485	2.787
26	0.127	0.256	0.473	0.684	1.315	1.706	2.056	2.479	2.779
27	0.127	0.256	0.473	0.684	1.314	1.703	2.052	2.473	2.771
28	0.127	0.256	0.473	0.683	1.313	1.701	2.048	2.467	2.763
29	0.127	0.256	0.473	0.683	1.311	1.699	2.045	2.462	2.756
30	0.127	0.256	0.472	0.683	1.310	1.697	2.042	2.457	2.750
40	0.126	0.255	0.471	0.681	1.303	1.684	2.021	2.423	2.704
50	0.126	0.255	0.471	0.679	1.299	1.676	2.009	2.403	2.678
60	0.126	0.254	0.470	0.679	1.296	1.671	2.000	2.390	2.660
70	0.126	0.254	0.470	0.678	1.294	1.667	1.994	2.381	2.648
80	0.126	0.254	0.469	0.678	1.292	1.664	1.990	2.374	2.639
90	0.126	0.254	0.469	0.677	1.291	1.662	1.987	2.368	2.632
100	0.126	0.254	0.469	0.677	1.290	1.660	1.984	2.364	2.626
∞	0.126	0.253	0.468	0.674	1.282	1.645	1.960	2.326	2.576